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SOME PROPERTIES OF SELF-SIMILAR CONVEX POLYTOPES

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Abstract. We show that for each semigroup G of similarities defining the self-similarity structure on a convex self-similar polytope K there is an edge A of K such that the fixed points of homotheties $g \in G$ are dense in A.

Keywords: self-similar set, fractal, convex polytope, graph-directed IFS, homothety, semigroup.

The interplay between the concepts of self-similarity and convexity is a promising and still unexplored field in the theory of self-similar fractals.

The first attempt to study the convex hulls of self-similar sets was made in 1993 by P. Panzone [1] who found the sufficient conditions for the self-similar set in $\mathbb{R}^n$ to have a finite polyhedral convex hull. In 1999, R. Strichartz and Y. Wang [2] obtained necessary and sufficient conditions for the finiteness of the convex hull for self-affine tiles in $\mathbb{R}^n$. The solution of this problems for self-affine multitiles was proposed in 2010 by I. Kirat and I. Kocyigit [3]. The use of the convex hulls and polyconvex prefractals was one of the main tools to investigate the curvature of self-similar and random self-similar sets in the recent works of S. Winter and M. Zahle [4, 5].

The simplest self-similar sets are line segments and convex polytopes. In the papers [6, 7] authors specified the conditions under which convex hull of a self-similar set in a Banach space is a finite polytope. However, the conditions for self-similar set to be itself a convex polytope in R^n and properties of self-similar structures on convex sets have not yet been studied. In this note we investigate the properties of homotheties in the semigroup $G(\mathcal{S})$ of similarities determining the self-similar structure on a finite polytope K.

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Notation.

1. Let $\mathcal{S} = \{S_1 ... S_m\}$ be a system of contraction similarities of $\mathbb{R}^n$. A nonempty compact set $K \subset \mathbb{R}^n$ is called an *invariant set* or an *attractor* of the system $\mathcal{S}$ if $K = \bigcup_{i=1}^{m} S_i(K)$. For each $x \in K$ there is such $S_i \in \mathcal{S}$, that $x \in S_i(K)$. The point $x_1 = S_i^{-1}(x)$ is called a *predecessor* of x. For each x there is a (possibly non-unique) sequence of predecessors $x_1, x_2, ...$ and a sequence of indices $i_1, i_2, ...$ such that for each k, $S_{k+1}(x_{k+1}) = x_k$. By $G(\mathcal{S})$ we denote the semigroup generated by the system $\mathcal{S}$.

2. Let $\Gamma = (V, E)$ be a directed multigraph with the set of vertices V and the set of edges E , each edge $e \in E$ having *the initial vertex* $\alpha(e) \in V$ and *the final vertex* $\omega(e) \in V$. The set of all edges $e \in E$, for which $\alpha(e) = u$ and $\omega(e) = v$ will be denoted by E_{uv}. Let $\{X_u, u \in V\}$ be a family of complete metric spaces and for each $e \in E_{uv}$, let $S_e : X_v \to X_u$ be a contraction similarity. Then the system $S_e, e \in E$ is called a *graph directed system of similarities* with a structural graph Γ. A family $\{K_u, u \in V\}$ of nonempty compact subsets $K_u \in X_u$ is called the *attractor* of the graph directed system $\mathcal{S}$, if for all $u \in V$, $K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} S_e(K_v)$.

3. Each contraction similarity $S : \mathbb{R}^n \to \mathbb{R}^n$ is uniquely represented in the form $S(x) = q \cdot O(x - a) + a$, where a is the fixed point of S, q is the contraction ratio, and O is the orthogonal transformation of $\mathbb{R}^n$, which we call the *orthogonal part* of S. If $O = \mathrm{Id}$, then S is called a *homothety.*

First we observe that if the invariant set K of a system $\mathcal{S}$ is a finite polytope, then there is an induced graph-directed system $\mathcal{S}'$ acting on the family of all k-faces of K:

Theorem 1. *Let $\mathcal{S}$ be a system of contraction similarities in $\mathbb{R}^n$, whose invariant set K is a finite convex polytope. For each $k = 0, 1, ..., n$, there is a finite set of k-faces $A_j^{(k)}$ of K such that the set of fixed points of the similarities $g \in G(\mathcal{S})$ sending $A_j^{(k)}$ into itself is dense in $A_j^{(k)}$.*

Proof. Denote the set of all k-faces of the polytope K by $F^{(k)}$. Let x be a point lying on k-face A_i of the polytope K. Then there is such k-face A' and a similarity $S' \in \mathcal{S}$, that $x \in S'(A') \subset A_i$.

Indeed, if x' is a predecessor of x and $S' \in \mathcal{S}$ satisfies $S'(x') = x$, then solid tangent cones $\mathcal{C}(x')$ and $\mathcal{C}(x)$ to K at points x' and x satisfy the inclusion $S'(\mathcal{C}(x')) \subset \mathcal{C}(x)$. Since the cone $\mathcal{C}(x)$ does not contain any $(k+1)$-plane in $\mathbb{R}^n$, the same is true for the cone $\mathcal{C}(x')$. Thus, x' belongs to some k-face A' of the polytope K and therefore A is contained in the union $\bigcup_{S_l \in \mathcal{S}} \bigcup_{A_j \in F^{(k)}} S_l(A_j)$ of the images $S_l'(A_j)$ of k-faces of K.

Since the union of all $(k-1)$-faces of the polytope K under similarities $S_i \in \mathcal{S}$ is nowhere dense in A, the k-face A is a subset of the union $\bigcup S_i'(A_j)$, where S_i' are only those elements of $\mathcal{S}$ for which the the intersection of the interiors of k-polytopes $S_i'(A_j)$ and A is nonempty. If this intersection is nonempty, the inclusion $S_i'(A_j) \subset K$ implies that $S_i'(A_j) \subset A$.

Let A_i, A_j be k-faces of K. Denote by $L(i,j)$ the set of all $l = 1, ..., m$, such that $S_l(A_j) \subset A_i$. Denote by S_{ijl} the restriction of S_l to the k-face A_j. Thus we define for each k-face A_i of the polytope K a finite set of similarities $S_{ijl}, l \in L(i,j)$ satisfying the following conditions:

1. The mapping S_{ijl} is a restriction of some similarity $S_l \in \mathcal{S}$ to the k-face A_j;
2. for each $l \in L(i,j)$, $S_{ijl}(A_j) \subset A_i$;
3. $A_i = \bigcup\limits_{A_j \in F^{(k)}} \bigcup\limits_{l \in L(i,j)} S_{ijl}(A_j)$

Let $\Gamma' = (F^{(k)}, E')$ be a directed multigraph with the set of vertices $F^{(k)}$ and the set of edges $E' = \bigcup\limits_{A_i, A_j \in F^{(k)}} E'_{ij}$, where E'_{ij} is the set of all triples $e = (i,j,l), l \in L(i,j)$. The conditions (1-3) mean that the system $\mathcal{S}' = \{S_{ijl}, l \in L(i,j)\}$ is a graph directed system of similarities with a structural graph Γ' and its attractor is just the family of all k-faces A_j.

The directed multigraph Γ' contains at least one strongly connected component Γ_0 without outgoing edges having their final vertices in the complement of Γ_0. Let $(F_0^{(k)}, E'_0)$ be the sets of vertices and edges of the subgraph Γ_0, and $\mathcal{S}'_0$ be the set of all similarities $S_e, e \in E'_0$. If $A_i \in F_0^{(k)}$ then for each $A_j \in F_0^{(k)}$ the set $L(i,j)$ defines the edges of the subgraph Γ_0 going from A_i to A_j; at he same time, if $A_j \notin F_0^{(k)}$, then the set $L(i,j)$ is empty. Therefore we have the equality $A_i = \bigcup\limits_{A_j \in F_0^{(k)}} \bigcup\limits_{l \in L(i,j)} S_{ijl}(A_j)$, which shows that for the graph-directed system $\mathcal{S}'_0$ its attractor is the family of all k-faces $A_j \in F_0^{(k)}$.

The set G_{A_j} of all those elements of the semigroup $G(\mathcal{S})$, which map a k-face $A_j \in F_0^{(k)}$ to itself, is a semigroup itself. As it was shown in [8], since the graph Γ_0 is strongly connected, the set of the fixed points z_g of the similarities $g \in G_{A_j}$ is dense in A_j. □

In dimension $k = 1$ and $k = 2$ it follows that some powers of the elements of semigroups G_{A_j} are homotheties:

Corollary 2. *There is a vertex z_0 of the polytope K, which is a fixed point of some homothety $g \in G(\mathcal{S})$.*

Proof. By the Theorem 1, there is a vertex z_0 of the polytope K, which is a fixed point of some similarity $g \in G(\mathcal{S})$. Let $g(x) = q \cdot O(x - z_0) + z_0$, where $q = \mathrm{Lip}(g)$ and $\mathcal{O}$ is the orthogonal part of g. The orthogonal transformation $\mathcal{O}$ maps the solid tangent cone $\mathcal{C}(z_0)$ to itself. The cone $\mathcal{C}(z_0)$ is a salient polyhedral convex cone. Therefore it is bounded by a finite number of support hyperplanes, and normal vectors to these hyperplanes contain a basis for $\mathbb{R}^n$. The action of $\mathcal{O}$ induces a permutation of these normal vectors. Then some power $\mathcal{O}^p$ fixes these normal vectors and therefore is equal to identity. Then g^p is a homothety. □

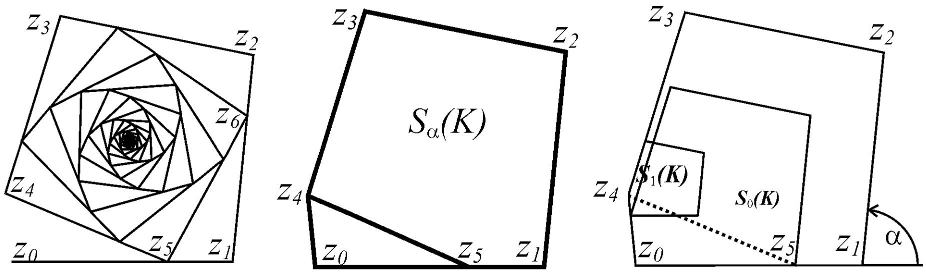
Corollary 3. *There is an edge A_j of the polytope K such that the set of fixed points of homotheties $g \in G(\mathcal{S})$ sending A_j into itself is dense in A_j.*

Proof. By the Theorem 1, there is an edge $A_j \in F^{(1)}$, for which the set of the fixed points z_g of the similarities $g \in G_{A_j}$ is dense in A_j. Consider such a similarity g and let O be it's orthogonal part. Let M be the hyperplane containing the point z_g and orthogonal to A_j and let $\mathcal{C}(z_g)$ be the solid tangent cone to K at the point z_g. Since $O(M) = M$ and $O(\mathcal{C}(z_g)) = \mathcal{C}(z_g)$, $O(\mathcal{C}(z_g) \cap M) = \mathcal{C}(z_g) \cap M$. Since

$\mathcal{C}(z_g) \cap M$ is a salient polyhedral convex cone in M, the argument of Corollary 2 shows that for some p, $\mathcal{O}^p|_M = \mathrm{Id}$. Then g^{2p} is a homothety. □

Corollary 4. *For each $k = 1, ..., n-1$ there is a k-face A_j of the polytope K, such that the set of fixed points z_g of those similarities $g \in G(\mathcal{S})$ sending A_j into itself, whose restriction to the orthogonal $(n-k)$-plane to $A-j$ at the point z_g is a homothety, is dense in A_j.*

Example 5. The following example shows that for a system $\mathcal{S}$ of similarities of $\mathbb{R}^2$, whose invariant set K is a convex polygon, the set of all fixed points z_g of homotheties $g \in G(\mathcal{S})$ may be contained in one of its sides.



Take such angle $\alpha \in (2\pi/5, \pi/2)$ that α/π is irrational. For $0 < q < 1$ define a similarity S_α in $\mathbb{C}$ by the formula $S_\alpha(z) = 1 + qe^{i\alpha}z$. Put $z_0 = 0$, $z_1 = 1$, $z_{k+1} = S_\alpha(z_k)$. Find such value of q, that $\mathrm{Im}(z_5) = 0$. Let K be a pentagon with vertices z_0, z_1, z_2, z_3, z_4. Define also the similarities $S_0(z) = z_5 z$ and $S_1(z) = (1-z_5)z + z_5 z_4$. Then $K = S_0(K) \cup S_1(K) \cup S_\alpha(K)$. Therefore, the polygon K is the invariant set of the system $\mathcal{S} = \{S_0, S_1, S_\alpha\}$. Consider the subsemigroup G' of the semigroup $G(\mathcal{S})$, generated by the transformations S_0 and S_1. All the elements $g \in G'$ are homotheties, and their fixed points form a dense subset of the line segment $[z_0, z_4]$. For each of the similarities $g \in G \setminus G'$ it's orthogonal part is a rotation to some positive integral multiple of the angle α. Therefore there are no any homothety in G whose fixed point is not contained in $[z_0, z_4]$.

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