S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 6, стр. 522–525 (2009) УДК 514.8,515.12
Краткие сообщения MSC 28A80

ON THE LENGTH OF THE SET OF EXTREME POINTS FOR SELF-SIMILAR SETS IN $\mathbb{R}^2$.

A.V. TETENOV

Abstract. We proof that the set of extreme points of the convex hull of any self-similar set in $\mathbb{R}^2$ has zero 1-dimensional Lebesgue measure.

Keywords: self-similar sets, fractal, convex hull, extreme points, Hausdorff measure.

1. Introduction

The interplay between the concepts of self-similarity and convexity is a promising and still unexplored field in the theory of self-similar fractals. This sharply differs from the situation in the theory of Kleinian groups, where the study of convex hulls of the limit sets (which are self-conformal fractals) for a long time serves as one of the main research tools in the theory.

The first attempt to study the convex hulls of self-similar sets was made in 1993 by P. Panzone [2] who found the sufficient conditions for the self-similar set in $\mathbb{R}^n$ to have a finite polyhedral convex hull. In 1999, R. Strichartz and Y. Wang [3] obtained necessary and sufficient conditions for the finiteness of the convex hull for self-affine tiles in $\mathbb{R}^n$. In 2006 M. Ferrari [1] considered the properties of the boundary points of the convex hulls of self-similar sets in $\mathbb{R}^3$. The use of the convex hulls and polyconvex prefractals was one of the main tools to investigate curvature of self-similar and random self-similar sets in the recent works of S. Winter and M. Zahle [7, 8].

Tetenov, A.V., On the length of the set of extreme points for self-similar sets in $\mathbb{R}^2$.

© 2009 Tetenov, A.V.

The work is supported by RFFI (grants 09-01-90202 and 09-01-98001) and ADTP (grant 2.1.1/3707).

Communicated by A.D. Mednykh December, 11, 2009, published December, 12, 2009.

It was shown by the author in 2002 [4, 5] that for the self-similar sets in $\mathbb{R}^2$ satisfying the open convex set condition, the set of the extreme points for a self-similar sets has zero Hausdorff dimension. In this paper we show that without any assumptions for a self-similar set in the plane the set F of the extreme points of it's convex hull has zero 1-dimensional Hausdorff measure and therefore the set F is a nowhere dense and totally disconnected compact subset of the boundary of the convex hull of the self-similar set.

Definitions. Let $\mathcal{S}$ be a system $\{S_1, ..., S_m\}$ of injective contraction similarities in $\mathbb{R}^2$ to itself and let K be it's *invariant set*, that is a non-empty compact set K satisfying $K = \bigcup_{i=1}^{m} S_i(K)$. The set K is also called the *attractor of the system* $\mathcal{S}$. Denote by G the semigroup of similarities generated by the system $\mathcal{S}$.

Consider *the convex hull* of the invariant set K, which we will denote by $\widetilde{K}$. Let F be *the set of all extreme points* of $\widetilde{K}$. Since $\widetilde{K}$ is the convex hull of F, the set F is contained in K.

Since the set $\widetilde{K}$ is a compact convex set in the plane, it's boundary $\partial\widetilde{K}$ is a bounded closed rectifiable curve and it's length is finite, therefore the one-dimensional Hausdorff measure of the set F is also finite.

The aim of this article is to prove the following theorem.

Theorem 1. *Let $\mathcal{S}$ be a system $\{S_1, ..., S_m\}$ of injective contraction similarities in $\mathbb{R}^2$ with invariant set K and F be the set of extreme points of the convex hull $\widetilde{K}$ of K. Then the one-dimensional Hausdorff measure of F is zero.*

2. The dynamics of the set of extreme points.

We recall some main properties of the set F of extreme points of a self-similar set K, which were proved in [4, 5, 6].

1. Let $z_0 \in F$ be the extreme point of $\widetilde{K}$. If for some $S_i \in \mathcal{S}$, $z_0 \in S_i(K)$, then z_0 is the extreme point of $S_i(K)$. Therefore $z_1 = S_i^{-1}(z_0)$ is also the extreme point of $\widetilde{K}$. For each $z_0 \in F$, there is at least one such z_1, and we call it *a predecessor* of z_0. By inductive reasoning we obtain an infinite sequence of extreme points $\{z_0, z_1, z_2, ...\}$, where z_{i+1} is the predecessor of z_i for each natural i.

2. An extreme point z_0 is called *a corner point* if the angle $\beta(z_0)$ between the right tangent ray to $\widetilde{K}$ at the point z_0 and the left one is less than π. It was proved in [4, 6] that for any corner point z_0, $\beta(z_1) < \beta(z_0)$ and the number of different points in the sequence of predecessors for the corner point z_0 is finite. This implies that each corner point is isolated in F and the set F_c of all corner points is at most countable. The set $G(F_c)$ of all images of corner points under the transformations of the semigroup G is also countable.

3. An extreme point z is called *an edge endpoint*, if there is such extreme point z' that the whole line segment $[z, z']$ is contained in the boundary $\partial\widetilde{K}$ of the convex hull $\widetilde{K}$. The segment $[z, z']$ will be called *the edge* of $\widetilde{K}$. Two extreme points $z, z' \in F$ are the endpoints of the same edge if and only if there is a normal vector to the boundary $\partial\widetilde{K}$ common for the points z and z'.

Since $\partial\widetilde{K}$ has finite length, the set of all edges of $\widetilde{K}$ is at most countable, and so is the set F_e of all edge endpoints. Let z be an edge endpoint and for some i the image $S_i(z)$ is an extreme point, then $S_i(z)$ is also the edge endpoint.

The following steps would prepare us to the proof of the Theorem 1.

4. Define $A = F \setminus (F_e \cup G(F_c))$. Since the set $(F_e \cup G(F_c))$ is at most countable, for each positive λ it's Hausdorff measure $H^\lambda(F_e \cup G(F_c))$ is zero, so it's enough to prove the conclusion of the Theorem 1 for the set A. Since F is compact, the set A is a Borel set.

Let $A_k = S_k(A) \setminus \bigcup_{i=1}^{k-1} S_i(A)$. The family of subsets $A_k, k = 1, ..., m$, is a partition of the set A to the Borel sets. For any $z \in A_k$, the point $S_k^{-1}(z) \in A$ is it's predecessor, and the map ${S_k}^{-1} : A_k \to A$ is an expanding similarity with the ratio ${q_i}^{-1}$, where $q_i = \mathrm{Lip}(S_i)$. Define the mapping $\psi : A \to A$ by the equality $\psi|_{A_k} = {S_k}^{-1}$.

5. Each point $z_0 \in A$ is not a corner point, so there is an unique outward unit normal vector $\nu(z_0)$, defining the map $\nu : A \to S^1$. Since A does not contain edge endpoints, for any two different points z, z' in A, $\nu(z) \neq \nu(z')$. Thus, the mapping $\nu : A \to S_1$ is injective.

Take a point $z \in A_k$. The angle between $\nu(z)$ and $\nu({S_k}^{-1}(z))$ is $-\alpha_k$, where α_k is the angle of rotation of the similarity S_k. Let r_k be the rotation in the angle $-\alpha_k$. Then the mapping ψ satisfies the equation $\nu \cdot \psi|_{A_k} = r_k \cdot \nu|_{A_k}$ for each k.

The main step in the proof of the Theorem 1 is the following

Lemma 2. *Let A be a Borel set in R^d and let $A = \bigcup_{i=1}^{n} A_i$ be it's partition into disjoint Borel subsets. Let $\psi : A \to A$ be such a map that for each i, the restriction $\psi_i = \psi|_{A_i}$ is an expanding similarity. Suppose there exist such an injective mapping $\nu : A \to S^1$ that for all $i = 1, ..., m$ and all $x \in A_i$, $\nu \cdot \psi(x) = r_i \cdot \nu(x)$, where r_i is a rotation of S^1 by angle α_i.*

Then for any $\lambda > 0$ the Hausdorff measure $H^\lambda(F)$ is either 0 or ∞.

Proof. Let $q_i = \mathrm{Lip}(\psi_i)^{-1}$ and $q = \min(q_1, ..., q_m)$.
Set $A_{i_1,..,i_{k+1}} = \psi^{-1}_{i_1,..,i_k}(\psi_{i_{k+1}}(A_{i_1,..,i_k} \cap A_{i_{k+1}}))$.

For each given k the family $\{A_\mathbf{i}, \mathbf{i} = i_1, ..., i_k \in I^k\}$ is a partition of the set A into disjoint Borel subsets.

Denote by $A(k, m)$ the set of all unordered arrays of the length k consisting of numbers $\{1, ..., m\}$. For each array $j \in A(k, m)$ we denote by $B(j)$ the set of all *ordered* arrays of length k, consisting of the elements of the array j.

Let $W_j^k = \bigcup_{\mathbf{i} \in B(j)} A_\mathbf{i}$.

For each k the sets W_j^k form a partition of the set A.

Each of the sets W_j^k satisfies the condition:

$$\text{for each } x \in W_j^k, \quad \pi \cdot \psi^k(x) = r_{\alpha(j)} \cdot \pi(x),$$

where $r_{\alpha(j)}$ — is a rotation by angle $\alpha(j) = \alpha_{i_1} + ... + \alpha_{i_k}$, where the indices are given by $i_1...i_k = \mathbf{i} \in B(j)$. This last sum does not depend from the order in which the indices i_k are taken but only of the array j and therefore is the same for all points $x \in W_j^k$. So for each $\mathbf{i}, \mathbf{i}' \in B(j)$, $\psi^k(U_\mathbf{i}) \cap \psi^k(U_{\mathbf{i}'}) = \emptyset$.

Thus, the set $\psi^k(W_j^k) \subset A$ is a finite union of disjoint subsets $\psi^k(A_\mathbf{i})$, therefore it's Hausdorff $\lambda-$measure H^λ satisfies the inequality

$$H^\lambda(\psi^k(W_j^k)) \leq H^\lambda(F).$$

Since for any set $A_{\mathbf{i}}, \mathbf{i} \in B(j)$, the restriction of ψ^k to the set $A_{\mathbf{i}}$ is an expanding similarity whose ratio does not exceed q^{-k},

$$H^\lambda(W_j^k) \leq q^{k\lambda} H^\lambda(\psi^k(W_j^k)) \leq q^{k\lambda} H^\lambda(A).$$

From the other side, the sets W_j^k partition of the set A, therefore

$$\sum_{j \in A(k,m)} H^\lambda(W_j^k) = H^\lambda(F).$$

As a result one obtains the inequality

$$H^\lambda(F) \leq q^{k\lambda} H^\lambda(F) \cdot \# A(k,m).$$

Since the set $A(k,m)$ consists of $\dfrac{(m+k-1)!}{(k-1)!m!}$ elements, and this number is no greater than k^m, for any k we obtain

$$H^\lambda(A) \leq k^m q^{k\lambda} H^\lambda(A).$$

This is possible only if $H^\lambda(A)$ is either equal to 0 or is infinite. ∎

The proof of the Theorem 1 is finished by the observation that 1-dimensional Hausdorff measure of $\partial\widetilde{K}$, and therefore of the set A is finite. Then by the Lemma 2, $H^1(A) = 0$. Therefore $H^1(F)$ is also zero. ∎

References

[1] Ferrari M., *The convex hull of self-similar sets in* $\mathbb{R}^3$, Fractals, **15**: 2 (2007), 1–10.

[2] Panzone P. A., *A note on the convex hulls of self-similar sets*, Actas del 2 congreso " Dr. A.A. Monteiro" Inst. de Matematica de Bahia Blanca, 1993, 57–64.

[3] R. Strichartz and Y. Wang, *Geometry of self-afiine tiles I.*, Indiana Univ. Math. J. 48 (1999), 1–24.

[4] Тетенов А. В., *Хаусдорфова размерность множества крайних точек самоподобных множеств на плоскости*, Динамика сплошной среды, Сб. науч. тр., Сиб. отд. РАН, Ин-т гидродинамики., Вып. 120 (2002), 53–59.

[5] Tetenov A. V., Davydkin I. B., *Hausdorff dimension of the set of extreme points of a self-similar set*, www.arXiv.org/math.MG/0202215.

[6] Тетенов А. В., Давыдкин И. Б., *О выпуклых оболочках самоподобных множеств*, Вестник новосибирского государственного университета Серия "Математика, механика, информатика", **V**: 2 (2005), 21–27.

[7] Winter S., *Curvature measures and fractals*, Diss. Math. 453, (2008), 66 pp.

[8] Zahle M., *Curvature densities of self-similar sets*, Workshop on Fractals and Tilings 2009 July 6–10, Strobl, Austria.

Andrey V. Tetenov,
Gorno-Altaisk state university,
Lenkin str.,1, 649000, Gorno-Altaisk, Russia
E-mail address: atet@mail.ru